

WHAT IS A LOW-ENERGY THEOREM ? ^{#1}

G. Ecker^{§,‡} and Ulf-G. Meißner^{†,*,#2}

[§]Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Wien,
Austria

[†]Physique Théorique, Centre de Recherches Nucléaires et Université Louis Pasteur de
Strasbourg, B.P. 28, F-67037 Strasbourg Cedex 2, France

email: [‡]ecker@ariel.pap.univie.ac.at, ^{*}meissner@crnvax.in2p3.fr

ABSTRACT

We discuss the meaning of low-energy theorems (LETs) in the framework of the effective field theory of the standard model. Particular emphasis is put on the LET for neutral pion photoproduction off nucleons at threshold. The seemingly controversial situation surrounding this LET is clarified.

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^{#2}Address after October 1st, 1994 : Universität Bonn, Institut für Theoretische Kernphysik, Nussallee 14-16, D-53115 Bonn, Germany

1 Introduction

Over the last few years, there has been some debate about the low-energy theorem (LET) for the electric dipole amplitude E_{0+} in the reaction $\gamma p \rightarrow \pi^0 p$ at threshold. This was spurred by the experimental findings that the LET derived in the early 1970's [1] [2] seemed to be violated [3] [4] leading to numerous re-examinations of the data as well as theoretical reconsiderations of the LET. A lucid discussion of the status as of 1991 can be found in the comment by Bernstein and Holstein [5]. In this comment, we wish to elaborate on certain aspects of LETs in the framework of the standard model (SM). In particular, we propose an answer to the question : "What is a LET ?".

Let us first consider a well-known example of a LET about which there is no discussion. Consider the scattering of very soft photons on the proton, i.e., the Compton scattering process $\gamma(k_1) + p(p_1) \rightarrow \gamma(k_2) + p(p_2)$ and denote by ε (ε') the polarization vector of the incoming (outgoing) photon. The transition matrix element T (normalized to $d\sigma/d\Omega = |T|^2$) can be expanded in a Taylor series in the small parameter $\delta = |\vec{k}_1|/m$, with m the nucleon mass. In the forward direction and in a gauge where the polarization vectors have only space components, T takes the form

$$T = c_0 \vec{\varepsilon}' \cdot \vec{\varepsilon} + i c_1 \delta \vec{\sigma} \cdot (\vec{\varepsilon}' \times \vec{\varepsilon}) + \mathcal{O}(\delta^2) . \quad (1)$$

The parameter δ can be made arbitrarily small in the laboratory so that the first two terms in the Taylor expansion (1) dominate. To be precise, the first one proportional to c_0 gives the low-energy limit for the spin-averaged Compton amplitude, while the second ($\sim c_1$) is of pure spin-flip type and can directly be detected in polarized photon proton scattering. The pertinent LETs fix the values of c_0 and c_1 in terms of measurable quantities [6],

$$c_0 = -\frac{Z^2 e^2}{4\pi m}, \quad c_1 = -\frac{Z^2 e^2 \kappa_p^2}{8\pi m} \quad (2)$$

with $Z = 1$ the charge of the proton and $\kappa_p = 1.793$ its anomalous magnetic moment. To arrive at Eq. (2), one only makes use of gauge invariance and the fact that the T -matrix can be written in terms of a time-ordered product of two conserved vector currents sandwiched between proton states. The derivation proceeds by showing that for small enough photon energies the matrix element is determined by the electromagnetic form factor of the proton at $q^2 = 0$ [6].

Similar methods can be applied to other than the electromagnetic currents. In strong interaction physics, a special role is played by the axial-vector currents. The associated symmetries are spontaneously broken giving rise to the Goldstone matrix elements

$$\langle 0 | A_\mu^a(0) | \pi^b(p) \rangle = i \delta_{ab} F_\pi p_\mu \quad (3)$$

where a, b are isospin indices and $F_\pi \simeq 93$ MeV is the pion decay constant. In the chiral limit (vanishing quark masses) the massless pions play a similar role as the photon

and many LETs have been derived for “soft pions”. In light of the previous discussion on Compton scattering, the most obvious one is Weinberg’s prediction for elastic πp scattering [7]. We only need the following translations :

$$< p | T j_\mu^{\text{em}}(x) j_\nu^{\text{em}}(0) | p > \rightarrow < p | T A_\mu^{\pi^+}(x) A_\nu^{\pi^-}(0) | p > , \quad (4)$$

$$\partial^\mu j_\mu^{\text{em}} = 0 \rightarrow \partial^\mu A_\mu^{\pi^-} = 0 . \quad (5)$$

In contrast to photons, pions are not massless in the real world. It is therefore interesting to find out how the LETs for soft pions are modified in the presence of non-zero pion masses (due to non-vanishing quark masses). In the old days of current algebra, a lot of emphasis was put on the PCAC (Partial Conservation of the Axial-Vector Current) relation, consistent with the Goldstone matrix element (3),

$$\partial^\mu A_\mu^a = M_\pi^2 F_\pi \phi_\pi^a , \quad (6)$$

where ϕ_π^a denotes the pion field and $M_\pi \simeq 140$ MeV is the pion mass. Although the precise meaning of (6) has long been understood [8], it does not offer a systematic method to calculate higher orders in the momentum and mass expansion of LETs. The derivation of non-leading terms in the days of current algebra and PCAC was more an art than a science, often involving dangerous procedures like off-shell extrapolations of amplitudes (see also Sect. 5).

The modern developments in this field have replaced the old notions by the effective field theory (EFT) of the SM incorporating all the symmetries of the SM including the spontaneously broken chiral symmetry. This framework to be sketched in Sect. 2 allows for a systematic expansion of amplitudes and Green functions in terms of momenta and meson masses. One recovers all the old LETs that are rightfully called theorems, but one does not reproduce some of the old results that were based on unjustified assumptions not valid in the SM. After the general definition of a LET in the new framework in Sect. 2 emphasizing the concept of chiral power counting, we briefly treat $\pi\pi$ scattering as a special example in Sect. 3. In Sect. 4 we reconsider the LET for Compton scattering in the framework of heavy baryon chiral perturbation theory. Our main concern, however, will be to clarify the status of the LETs for E_{0+} in $\gamma N \rightarrow \pi^0 N$ at threshold (see also the discussion in Ref. [9]) in Sect. 5 and to discuss some of the pitfalls of the old methods that can be avoided with modern techniques.

2 Definition of low-energy theorems

Chiral perturbation theory (CHPT) is the EFT of the SM at low energies in the hadronic sector. Since as an EFT it contains all terms allowed by the symmetries of the underlying theory [10], it should be viewed as a direct consequence of the SM itself. The two main assumptions underlying CHPT are that

- (i) the masses of the light quarks u, d (and possibly s) can be treated as perturbations (i.e., they are small compared to a typical hadronic scale of 1 GeV) and that
- (ii) in the limit of zero quark masses, the chiral symmetry is spontaneously broken to its vectorial subgroup. The resulting Goldstone bosons are the pseudoscalar mesons (pions, kaons and eta).

CHPT is a systematic low-energy expansion around the chiral limit [10] [11] [12] [13]. It is a well-defined quantum field theory although it has to be renormalized order by order. Beyond leading order, one has to include loop diagrams to restore unitarity perturbatively. Furthermore, Green functions calculated in CHPT at a given order contain certain parameters that are not constrained by the symmetries, the so-called low-energy constants (LECs). At each order in the chiral expansion, those LECs have to be determined from phenomenology (or can be estimated with some model dependent assumptions). For a review of the wide field of applications of CHPT, see, e.g., Ref. [14].

In the baryon sector, a complication arises from the fact that the baryon mass m does not vanish in the chiral limit [15]. Stated differently, only baryon three-momenta can be small compared to the hadronic scale. To restore the correspondence between the loop and the energy expansion valid in the meson sector, one can reformulate baryon CHPT [15] in analogy to heavy quark effective theory to shift the troublesome mass term from the baryon propagator to a string of interaction vertices with increasing powers of $1/m$ [16]. The procedure is reminiscent of the well-known Foldy-Wouthuysen transformation and is called heavy baryon chiral perturbation theory (HBCHPT). The baryon four-momentum is written as $p_\mu = mv_\mu + l_\mu$, with v_μ the four-velocity and l_μ a small off-shell momentum, $v \cdot l \ll m$. The Dirac equation for the velocity-dependent baryon field B_v takes the form $iv \cdot \partial B_v = 0$ to lowest order in $1/m$. This allows for a consistent chiral counting as described below.

We are now ready to address the central question of this comment :

What is a LET ?

$$\begin{aligned} & \mathbf{L}(\mathbf{O})\mathbf{W} \mathbf{E}(\mathbf{N})\mathbf{E}(\mathbf{R})\mathbf{G}\mathbf{Y} \mathbf{T}(\mathbf{H})\mathbf{E}(\mathbf{O})\mathbf{R}(\mathbf{E})\mathbf{M} \text{ OF } \mathcal{O}(p^n) \\ & \equiv \text{GENERAL PREDICTION OF CHPT TO } \mathcal{O}(p^n) \end{aligned}$$

As will be explained below, p stands for a small momentum or mass characterizing the chiral expansion. By general prediction we mean a strict consequence of the SM depending on some LECs like $F_\pi, m, g_A, \kappa_p, \dots$, but without any model assumption for these parameters. This definition contains a precise prescription how to obtain higher-order corrections to leading-order LETs and it should therefore be generally adopted for hadronic processes at low energies. Although we have formulated the procedure with the SM in mind, the obtained LETs are actually more general. Since one only uses the symmetries of the SM to derive general results of CHPT, those results hold in fact in any theory that shares the symmetries of the SM. This general aspect of a LET is less relevant today than 30 years ago, but it should be kept in mind.

We have to be a little more precise what is meant by a result of $\mathcal{O}(p^n)$. From the outset, one can distinguish between an expansion in momenta (CHPT is a low-energy effective theory) and an expansion around the chiral limit in terms of quark masses. These two expansions become related by expressing the pseudoscalar meson masses in terms of the quark masses. We adopt here the standard assumption supported by the success of the Gell-Mann–Okubo mass formula for the pseudoscalar octet that the dominant contributions to the squares of the meson masses are linear in the quark masses, e.g.,

$$M_{\pi^+}^2 = B(m_u + m_d)[1 + \mathcal{O}(m_{quark})] . \quad (7)$$

The constant B is related to the quark condensate and is assumed to be non-vanishing in the chiral limit (supported by lattice data). In this case, Eq. (7) implies the standard chiral counting where quark masses count as $\mathcal{O}(p^2)$. If one declares the Gell-Mann–Okubo formula to be a numerical accident, one can envisage a situation where B is very small or even zero so that the higher-order terms in (7) could be dominant. The proponents of “Generalized CHPT” [17] account for this possibility by considering the quark masses as objects of $\mathcal{O}(p)$. In practice, this means that at any given order the CHPT generalizers include some additional terms which would only appear in higher orders in the standard counting. Since there is at this time no phenomenological necessity to include those terms (with their associated unknown LECs), we stick to the standard procedure. Of course, a difference can only appear in LETs where symmetry breaking terms in the chiral Lagrangian contribute. Anticipating the examples discussed below, the generalized counting affects $\pi\pi$ scattering already at $\mathcal{O}(p^2)$ (Sect. 3), but it does not modify the LETs for Compton scattering (Sect. 4) or for neutral pion photoproduction (Sect. 5).

The soft-photon theorems, e.g., for Compton scattering [6], involve the limit of small photon momenta, with all other momenta remaining fixed. Therefore, they hold to all orders in the non-photon momenta and masses. In the low-energy expansion of CHPT, on the other hand, the ratios of all small momenta and pseudoscalar meson masses are held fixed. Of course, the soft-photon theorems are also valid in CHPT as in any gauge invariant quantum field theory. We shall come back to this difference of low-energy limits in Sect. 4 in the derivation of the LET for Compton scattering.

To calculate a LET to a given order, it is useful to have a compact expression for the chiral power counting [10] [18]. We restrict ourselves to purely mesonic or single-nucleon processes. Any amplitude for a given physical process has a certain **chiral dimension** D which keeps track of the powers of external momenta and meson masses. The building blocks to calculate this chiral dimension from a general Feynman diagram in the CHPT loop expansion are

- (i) I_M meson propagators $\sim 1/(k^2 - M^2)$ (with M the meson mass) of dimension $D = -2$,
- (ii) I_B baryon propagators $\sim 1/v \cdot k$ (in HBCHPT) with $D = -1$,
- (iii) N_d^M mesonic vertices with $d = 2, 4, 6, \dots$ and
- (iv) N_d^{MB} meson-baryon vertices with $d = 1, 2, 3, \dots$

Putting these together, the chiral dimension D of a given amplitude reads

$$D = 4L - 2I_M - I_B + \sum_d d(N_d^M + N_d^{MB}) \quad (8)$$

with L the number of loops. For connected diagrams, we can use the general topological relation

$$L = I_M + I_B - \sum_d (N_d^M + N_d^{MB}) + 1 \quad (9)$$

to eliminate I_M :

$$D = 2L + 2 + I_B + \sum_d (d-2)N_d^M + \sum_d (d-2)N_d^{MB} . \quad (10)$$

Lorentz invariance and chiral symmetry demand $d \geq 2$ for mesonic interactions and thus the term $\sum_d (d-2)N_d^M$ is non-negative. Therefore, in the absence of baryon fields, Eq. (10) simplifies to [10]

$$D = 2L + 2 + \sum_d (d-2)N_d^M \geq 2L + 2 . \quad (11)$$

To lowest order p^2 , one has to deal with tree diagrams ($L = 0$) only. Loops are suppressed by powers of p^{2L} .

The other case of interest for us has a single baryon line running through the diagram (i.e., there is exactly one baryon in the in- and one baryon in the out-state). In this case, the identity

$$\sum_d N_d^{MB} = I_B + 1 \quad (12)$$

holds leading to [18]

$$D = 2L + 1 + \sum_d (d-2)N_d^M + \sum_d (d-1)N_d^{MB} \geq 2L + 1 . \quad (13)$$

Therefore, tree diagrams start to contribute at order p and one-loop graphs at order p^3 . Obviously, the relations involving baryons are only valid in HBCHPT.

Let us now consider diagrams with N_γ external photons.^{#3} Since gauge fields like the electromagnetic field appear in covariant derivatives, their chiral dimension is obviously $D = 1$. We therefore write the chiral dimension of a general amplitude with N_γ photons as

$$D = D_L + N_\gamma , \quad (14)$$

where D_L is the degree of homogeneity of the (Feynman) amplitude A as a function of external momenta (p) and meson masses (M) in the following sense ^{#4}:

$$A(p, M; C_i^r(\mu), \mu/M) = M^{D_L} A(p/M, 1; C_i^r(\mu), \mu/M) , \quad (15)$$

where μ is an arbitrary renormalization scale and $C_i^r(\mu)$ denote renormalized LECs. From now on, we suppress the explicit dependence on the renormalization scale and on the LECs. Since the total amplitude is independent of the arbitrary scale μ , one may in particular choose $\mu = M$. Note that $A(p, M)$ has also a certain physical dimension (which is of course independent of the number of loops and is therefore in general different from D_L). The correct physical dimension is ensured by appropriate factors of F_π and m in the denominators as will become evident from the following examples.

In the remaining sections, we always consider chiral $SU(2)$ in the isospin limit ($m_u = m_d$).

3 Pion–pion scattering

We first consider the mesonic sector. The purest reaction to test chiral dynamics is elastic $\pi\pi$ scattering in the threshold region. It involves exclusively Goldstone bosons and the expansion parameters $E^2/(4\pi F_\pi)^2$ and $M_\pi^2/(4\pi F_\pi)^2 \simeq 0.014$ are small. The $\pi\pi$ scattering amplitude can be written in terms of a single invariant function, conventionally called $A(s, t, u)$ where s, t and u are the Mandelstam variables. The chiral expansion of $A(s, t, u)$ takes the form

$$A(s, t, u) = A^{(2)}(s, t, u) + A^{(4)}(s, t, u) + \mathcal{O}(p^6)$$

where p^6 denotes terms of the type $E^6, E^4 M_\pi^2, E^2 M_\pi^4$ or M_π^6 ($E^2 = s, t$ or u). Since no external photons are involved, we have $D = D_L$. From Eq. (11) we read off that to lowest order $L = 0$ and $d = 2$ only (tree diagrams with insertions from the lowest-order chiral Lagrangian \mathcal{L}_2). The corresponding LET of order p^2 was derived by Weinberg [20] :

$$A^{(2)}(s, t, u) = \frac{s - M_\pi^2}{F_\pi^2} = \frac{M_\pi^2}{F_\pi^2} \left(\frac{s}{M_\pi^2} - 1 \right) . \quad (16)$$

^{#3}We remind the reader that CHPT as discussed here has **only** external photons.

^{#4}A similar observation has been made by Rho [19] in the context of meson exchange currents.

We notice that the degree of homogeneity $D_L = 2$ indeed differs from the physical dimension of the amplitude which in this case is dimensionless. If one projects the amplitude (16) onto channels with isopin I and angular momentum l of the two-pion system and expands the corresponding partial waves $t_l^I(s)$ in powers of the pion three-momentum, one can read off the well-known LET for the $I = 0$ S -wave scattering length [20] :

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16 , \quad (17)$$

to be compared with the empirical value of $a_{0,\text{exp}}^0 = 0.26 \pm 0.05$ [21].

At next-to-leading order, $D = D_L = 4$, one has two types of contributions. First, there are (divergent) loop diagrams with $L = 1$ and $N_d^M = 0$ for $d > 2$ and second, counterterms with $L = 0$, $N_4^M = 1$ and $N_d^M = 0$ ($d > 4$). The latter involve some LECs denoted \bar{l}_i in the $SU(2)$ analysis [11]. The complete amplitude of $\mathcal{O}(p^4)$ is of the form

$$A^{(4)}(s, t, u) = \frac{M_\pi^4}{F_\pi^4} \hat{A} \left(\frac{s}{M_\pi^2}, \frac{t}{M_\pi^2}, \frac{u}{M_\pi^2} \right) . \quad (18)$$

The LET of order p^4 for the $\pi\pi$ scattering amplitude is simply the sum of eqs. (16) and (18). In particular, the LET for a_0^0 to order p^4 reads [11]

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} \left\{ 1 + \frac{M_\pi^2}{3} \langle r^2 \rangle_S^\pi - \frac{M_\pi^2}{672\pi^2 F_\pi^2} (15\bar{\ell}_3 - 353) \right\} + \frac{25}{4} M_\pi^4 (a_2^0 + 2a_2^2) = 0.20 \pm 0.01 \quad (19)$$

with $\langle r^2 \rangle_S^\pi \simeq 0.6 \text{ fm}^2$ the scalar radius of the pion [22] and a_2^0, a_2^2 the D -wave scattering lengths. An estimate for the LEC $\bar{\ell}_3$ can be found in Ref. [11]. The precise prediction (19) awaits an equally accurate empirical determination (for a more detailed discussion on the relevance of pinning down a_0^0 , see, e.g., Ref. [14]).

4 Compton scattering revisited

In this section, we rederive and extend the LET for spin-averaged nucleon Compton scattering in the framework of HBCHPT [23]. Consider the spin-averaged Compton amplitude in forward direction (in the Coulomb gauge $\varepsilon \cdot v = 0$)

$$e^2 \varepsilon^\mu \varepsilon^\nu \frac{1}{4} \text{Tr} \left[(1 + \gamma_\lambda v^\lambda) T_{\mu\nu}(v, k) \right] = e^2 \left[\varepsilon^2 U(\omega) + (\varepsilon \cdot k)^2 V(\omega) \right] \quad (20)$$

with $\omega = v \cdot k$ (k is the photon momentum) and

$$T_{\mu\nu}(v, k) = \int d^4k \, e^{ik \cdot x} \langle N(v) | T j_\mu^{\text{em}}(x) j_\nu^{\text{em}}(0) | N(v) \rangle . \quad (21)$$

All dynamical information is contained in the functions $U(\omega)$ and $V(\omega)$. We only consider $U(\omega)$ here and refer to Ref. [23] for the calculation of both $U(\omega)$ and $V(\omega)$. In the Thomson limit, only $U(0)$ contributes to the amplitude.

In the forward direction, the only quantities with non-zero chiral dimension are ω and M_π . In order to make this dependence explicit, we write $U(\omega, M_\pi)$ instead of $U(\omega)$. With $N_\gamma = 2$ external photons, the degree of homogeneity D_L for a given CHPT contribution to $U(\omega, M_\pi)$ follows from Eq. (13) :

$$D_L = 2L - 1 + \sum_d (d - 2) N_d^M + \sum_d (d - 1) N_d^{MB} \geq -1 . \quad (22)$$

Therefore, the chiral expansion of $U(\omega, M_\pi)$ takes the following general form :

$$U(\omega, M_\pi) = \sum_{D_L \geq -1} \omega^{D_L} f_{D_L}(\omega/M_\pi) . \quad (23)$$

The following arguments illuminate the difference and the interplay between the soft-photon limit and the low-energy expansion of CHPT. Let us consider first the leading terms in the chiral expansion (23) :

$$U(\omega, M_\pi) = \frac{1}{\omega} f_{-1}(\omega/M_\pi) + f_0(\omega/M_\pi) + \mathcal{O}(p^3) . \quad (24)$$

Eq. (22) tells us that only tree diagrams can contribute to the first two terms. However, the relevant tree diagrams shown in Fig. 1 do not contain pion lines. Consequently, the functions f_{-1} , f_0 cannot depend on M_π and are therefore constants. Since the soft-photon theorem [6] requires $U(0, M_\pi)$ to be finite, f_{-1} must actually vanish and the chiral expansion of $U(\omega, M_\pi)$ can be written as

$$U(\omega, M_\pi) = f_0 + \sum_{D_L \geq 1} \omega^{D_L} f_{D_L}(\omega/M_\pi) . \quad (25)$$

But the soft-photon theorem yields additional information : since the Compton amplitude is independent of M_π in the Thomson limit and since there is no term linear in ω in the spin-averaged amplitude, we find

$$\lim_{\omega \rightarrow 0} \omega^{n-1} f_n(\omega/M_\pi) = 0 \quad (n \geq 1) \quad (26)$$

implying in particular that the constant f_0 describes the Thomson limit :

$$U(0, M_\pi) = f_0 . \quad (27)$$

Let us now verify these results by explicit calculation. In the Coulomb gauge, there is no direct photon-nucleon coupling from the lowest-order effective Lagrangian $\mathcal{L}_{\pi N}^{(1)}$ since it is proportional to $\varepsilon \cdot v$. Consequently, the Born diagrams a,b in Fig. 1 vanish so that

indeed $f_{-1} = 0$. On the other hand, the expansion of the relativistic Dirac Lagrangian leads to terms of the type $D^2/2m$ and $(v \cdot D)^2/2m$ where D_μ is a covariant derivative. Notice that although these terms belong to $\mathcal{L}_{\pi N}^{(2)}$, they do not contain novel LECs since they are of purely kinematical origin. These terms lead to a Feynman insertion (Fig. 1c) of the form

$$i \frac{e^2}{m} \frac{1}{2} (1 + \tau_3) \left[\varepsilon^2 - (\varepsilon \cdot v)^2 \right] = i \frac{e^2 Z^2}{m} \varepsilon^2 \quad (28)$$

producing the desired result $f_0 = Z^2/m$, the Thomson limit.

At the next order in the chiral expansion, $\mathcal{O}(p^3)$ ($D_L = 1$), the function $f_1(\omega/M_\pi)$ is given by the finite sum of 9 one-loop diagrams [24] [23]. According to Eq. (26), f_1 vanishes for $\omega \rightarrow 0$. The term linear in ω/M_π yields the leading contribution to the sum of the electric and magnetic polarizabilities of the nucleon, defined by the second-order Taylor coefficient in the expansion of $U(\omega, M_\pi)$ in ω :

$$f_1(\omega/M_\pi) = -\frac{11g_A^2\omega}{192\pi F_\pi^2 M_\pi} + \mathcal{O}(\omega^2) , \quad (29)$$

where g_A is the nucleon axial-vector coupling constant. The $1/M_\pi$ behaviour should not come as a surprise – in the chiral limit the pion cloud becomes long-ranged (instead of being Yukawa-suppressed) so that the polarizabilities explode. This behaviour is specific to the leading contribution of $\mathcal{O}(p^3)$. In fact, from the general form (25) one immediately derives that the contribution of $\mathcal{O}(p^n)$ ($D_L = n - 2$) to the polarizabilities is of the form $c_n M_\pi^{n-4}$ ($n \geq 3$), where c_n is a constant that may be zero.

One can perform a similar analysis for the amplitude $V(\omega)$ and for the spin-flip amplitude. We do not discuss these amplitudes here but refer the reader to Ref. [23] for details.

5 Neutral pion photoproduction at threshold

We consider the processes

$$\gamma N \rightarrow \pi^0 N \quad (N = p, n)$$

at threshold, i.e., for vanishing three-momentum of the pion in the nucleon rest frame. At threshold, only the electric dipole amplitude E_{0+} survives and the only quantity with non-zero chiral dimension is M_π . In the usual conventions, E_{0+} has physical dimension -1 and it can therefore be written as

$$E_{0+} = \frac{eg_A}{F} A \left(\frac{M_\pi}{m}, \frac{M_\pi}{F} \right) , \quad (30)$$

where F is the pion decay constant in the chiral limit. The dimensionless amplitude A will be expressed as a power series in M_π . The various parts are characterized by the

degree of homogeneity (in M_π) D_L according to the chiral expansion. Since $N_\gamma = 1$ in the present case, we obtain from Eq. (13)

$$D_L = D - 1 = 2L + \sum_d (d-2)N_d^M + \sum_d (d-1)N_d^{MB} . \quad (31)$$

For the LET of $\mathcal{O}(p^3)$ in question, only lowest-order mesonic vertices ($d = 2$) will appear. Therefore, in this case the general formula for D_L takes the simpler form

$$D_L = 2L + \sum_d (d-1)N_d^{MB} . \quad (32)$$

We now discuss the chiral expansion of E_{0+} step by step, referring to the literature [25] [26] [23] for the actual calculation.

$$\mathbf{D}_L = 0$$

From Eq. (32) we conclude that only tree diagrams ($L = 0$) with vertices from the $\mathcal{O}(p)$ chiral pion-nucleon Lagrangian $\mathcal{L}_{\pi N}^{(1)}$ can contribute. At threshold, the only diagram is the Kroll-Ruderman contact term [27] where both the pion and the photon emanate from the same vertex. However, this vertex only exists for charged pions. Thus, there is no term with $D_L = 0$ for neutral pion production.

$$\mathbf{D}_L = 1$$

In HBCHPT, the relevant tree diagram (remember that $L = 0$ for $D_L < 2$) looks exactly like the Kroll-Ruderman diagram, except that now the vertex comes from the $\mathcal{O}(p^2)$ pion-nucleon Lagrangian. In the relativistic formulation [15], the contribution is due to the normal scattering (and crossed) diagrams retaining only the nucleon mass in the nucleon propagator. HBCHPT replaces these diagrams by a contact term proportional to $1/m$. From the relativistic description it is clear that this contribution is proportional to the nucleon charge. For the neutron, both the $D_L = 0$ and the $D_L = 1$ pieces vanish.

$$\mathbf{D}_L = 2$$

The master formula (32) allows in principle for three types of contributions :

- (a) Tree level ($L = 0$) diagrams with a single vertex of $\mathcal{O}(p^3)$ ($N_3^{MB} = 1$, but all other $N_d^{MB} = 0$ for $d > 1$). Although such vertices exist, they can be shown not to contribute to neutral pion photoproduction at threshold [23]. This has another important implication : the loop contribution to be discussed below must be finite because chiral and gauge invariance do not permit appropriate counterterms of $\mathcal{O}(p^3)$.

- (b) There are non-vanishing Born diagrams with a nucleon propagator between $\mathcal{O}(p^2)$ γNN and πNN couplings, respectively ($L = 0$, $N_2^{MB} = 2$, $N_d^{MB} = 0$ ($d > 2$)). The γNN coupling is proportional to a LEC of HBCHPT, the magnetic moment of the nucleon (in the chiral limit).
- (c) Finally, and this is the piece that has generated a considerable amount of paper and some heated discussions, there is a one-loop contribution ($L = 1$) with leading-order vertices only ($N_d^{MB} = 0$ ($d > 1$)). It is considerably easier to work out the relevant diagrams in HBCHPT [23] than in the original derivation [25] [26]. In fact, at threshold only the so-called triangle diagrams shown in Fig. 2 survive out of some 60 diagrams. The main reason for the enormous simplification in HBCHPT is that one can choose a gauge without a direct γNN coupling of lowest order and that there is no direct coupling of the produced π^0 to the nucleon at threshold. As already noted, the loop contributions are finite and they are identical for proton and neutron. They were omitted in the original version of the LET [1] [2] and in many later rederivations.

The full LETs of $\mathcal{O}(p^3)$ are given by [25]

$E_{0+}(\pi^0 p) =$	$-\frac{eg_A}{8\pi F_\pi} \left[\begin{array}{cccc} 0 & + & \frac{M_\pi}{m} & - & \frac{M_\pi^2}{2m^2} (3 + \kappa_p) & - & \frac{M_\pi^2}{16F_\pi^2} & + \mathcal{O}(M_\pi^3) \end{array} \right]$
$D_L :$	$\begin{array}{cccc} 0 & 1 & 2_b & 2_c \end{array}$
$E_{0+}(\pi^0 n) =$	$-\frac{eg_A}{8\pi F_\pi} \left[\begin{array}{cccc} 0 & + & 0 & + & \frac{M_\pi^2}{2m^2} \kappa_n & - & \frac{M_\pi^2}{16F_\pi^2} & + \mathcal{O}(M_\pi^3) \end{array} \right]$

Two comments are in order here :

- (i) There is a kinematical factor in the relation between the electric dipole amplitude and the Feynman amplitude depending on M_π . Expanding this factor in M_π/m affects the $\mathcal{O}(M_\pi^2)$ term in the case of the proton. This explains the factor $3 + \kappa_p$ instead of $1 + \kappa_p$. For the neutron, this does not influence the LET to $\mathcal{O}(M_\pi^2)$ because there is no term with $D_L = 1$.
- (ii) All LECs appearing in the LETs are the physical quantities g_A , m , F_π , κ_p , κ_n although the effective chiral Lagrangian contains the corresponding quantities in the chiral limit. It is a major conceptual advantage of HBCHPT that the relation between the physical and the chiral limit values of all these parameters is such that

the differences can only appear in the higher-order terms denoted as $\mathcal{O}(M_\pi^3)$ (see also below). To prove the analogous statement in the relativistic formulation is much more cumbersome. In fact, most of the loop contributions encountered in the relativistic approach renormalize the various constants to their physical values [25] [26]. Of course, the final result is the same in both approaches.

The derivation of LETs sketched above is based on a well-defined quantum field theory where each step can be checked explicitly. Nevertheless, the corrected LETs have been questioned by several authors. We find it instructive to discuss some of the arguments and assumptions that have been used to derive or rederive the original LETs. Generically, those derivations are based on more or less plausible assumptions that qualify the results as LEGs (low-energy guesses) rather than LETs. Since we have CHPT at our disposal as the effective low-energy representation of the SM, we can actually check whether or not those assumptions hold in the SM. The following list is not meant to be exhaustive nor is it intended to be a compilation of mistakes in the published literature. It should rather be viewed as a collection of pitfalls that should be looked out for when extending LETs beyond leading order.

(a) Analyticity assumption

The original derivations [1] [2] and some later rederivations [28] [29] [30] [5] used a Taylor expansion of amplitudes in the variables ν, ν_B (linear combinations of the usual Mandelstam variables s and u). The seemingly plausible assumption that the coefficients of this expansion are analytic in M_π leads directly to the original LEG. In fact, in Ref. [1] it was explicitly spelled out that this is a necessary assumption for the LEG to hold. However, as shown in Ref. [25], this assumption does not hold in QCD. Due to the Goldstone nature of the pion, some Taylor coefficients diverge in the chiral limit. This happens precisely in the loop contributions ($D_L = 2$) which generate infrared divergences in some coefficients. The threshold amplitude itself is perfectly well-behaved in the chiral limit.

(b) External versus internal pion mass

It has been suggested [29] [31] [32] that there is a basic difference between the external, kinematical pion mass M_π and the internal mass \bar{M}_π appearing in the pion propagators in loop diagrams. The assumption is that \bar{M}_π appears only in relations between unrenormalized and renormalized quantities. Therefore, expressing everything in measurable, renormalized quantities, no trace of \bar{M}_π is left and one recovers the original LEG since the loop contribution is to be dropped by assumption.

Let us investigate this assumption in detail within HBCHPT. Denoting unrenormalized quantities (the parameters in the effective chiral Lagrangian) with a superscript \circ , one

finds the following relations between physical, renormalized quantities and their unrenormalized counterparts :

$$Q = \overset{\circ}{Q} [1 + \mathcal{O}(m_q)] \equiv \overset{\circ}{Q} [1 + \mathcal{O}(p^2)], \quad Q = M_\pi, F_\pi, m, g_A \quad (33)$$

$$\kappa = \overset{\circ}{\kappa} [1 + \mathcal{O}(m_q^{1/2})] \equiv \overset{\circ}{\kappa} [1 + \mathcal{O}(p)] . \quad (34)$$

As already emphasized before, renormalization in the framework of HBCHPT can therefore only affect terms of $\mathcal{O}(M_\pi^3)$ in the LETs for π^0 photoproduction at threshold. Thus, the loop contribution to the LETs cannot be a renormalization effect.

There is a more fundamental objection to the distinction between external and internal pion masses [33]. QCD does not offer a consistent procedure for $M_\pi \rightarrow 0$ with \bar{M}_π remaining finite. The only tunable mass parameters in QCD are the quark masses. Letting the quark masses tend to zero makes all pion masses vanish, whether they be external or internal.

(c) Off-shell expansion

The inadmissible distinction between external and internal pion masses can also appear in an off-shell extrapolation of the amplitude. Davidson has contrasted the expansion in M_π with a so-called ω expansion [34]. Keeping M_π fixed, he sets the three-momentum $\vec{p}_\pi = 0$ and expands in the pion energy $E_\pi = \omega$. Obviously, for $\omega \neq M_\pi$ this implies an off-shell extrapolation of the scattering amplitude. If one expands the amplitude first to $\mathcal{O}(\omega^2)$, the coefficients still depend on M_π . Expanding those coefficients in a second step in M_π so that the overall order is $\mathcal{O}(M_\pi^2)$ for $\omega = M_\pi$, one obtains the original LEG [34]. The mathematical origin of the problem is an illicit interchange of limits : expanding a function $f(\omega, M_\pi)$ in the manner just described and setting $\omega = M_\pi$ at the end will in general not lead to the same result as an expansion of $f(M_\pi, M_\pi)$ to the same order in M_π .

Although it is shown in Ref. [34] that one can recover the correct LET by a resummation of the series to all orders in ω , there is in general no guarantee that off-shell manipulations produce the correct result. A simple, but instructive example is to consider the elastic $\pi\pi$ scattering amplitude to lowest order, $\mathcal{O}(p^2)$, both in CHPT and in the linear σ model. Although the amplitudes agree on-shell, they disagree in general off-shell. In fact, one can obtain very different forms for the off-shell amplitude by redefining the pion field. While one would normally not employ such redefinitions in the linear model (seemingly destroying renormalizability), any choice of pion field is equally acceptable in CHPT which is based on an intrinsically non-renormalizable quantum field theory.

Off-shell manipulations are dangerous and may lead to incorrect results. The literature on applications of current algebra techniques abounds with examples.

(d) Phenomenology

Although the purpose of this comment is not to discuss the experimental situation, it may be one of nature's follies that experiments seem to favour the original LEG over the correct LET. One plausible explanation for the seeming failure of the LET is the very slow convergence of the expansion in M_π [26] [35]. CHPT produces a satisfactory description of the total and differential cross sections near threshold [3] [4] [26] [35]. On the other hand, the extrapolation to threshold involves sizable isospin violating corrections that are not fully under control [36]. Both for the isospin violating corrections and for the slow convergence of the expansion in M_π , the LET for π^0 photoproduction at threshold does not seem to be the ideal place to test the SM. It will however remain an important theoretical check for any model of hadronic interactions at low energies.

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Figure Captions

Fig. 1: Tree diagrams of $\mathcal{O}(p)$ (a,b) and $\mathcal{O}(p^2)$ (c) for Compton scattering in HBCHPT. Full (wavy) lines stand for nucleons (photons).

Fig. 2: One-loop triangle diagrams contributing to the threshold amplitude E_{0+} for π^0 photoproduction at $\mathcal{O}(M_\pi^2)$. Pions are denoted by broken lines.

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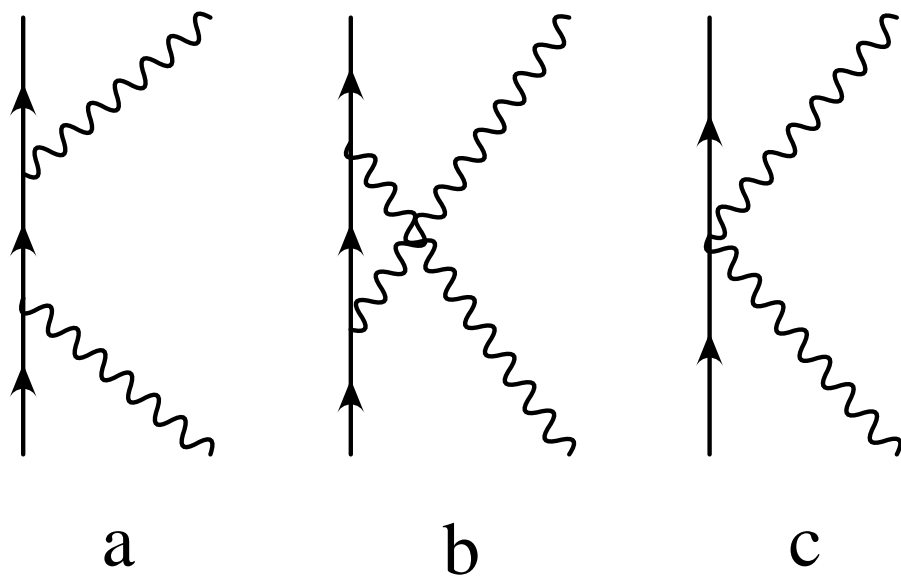


Fig. 1

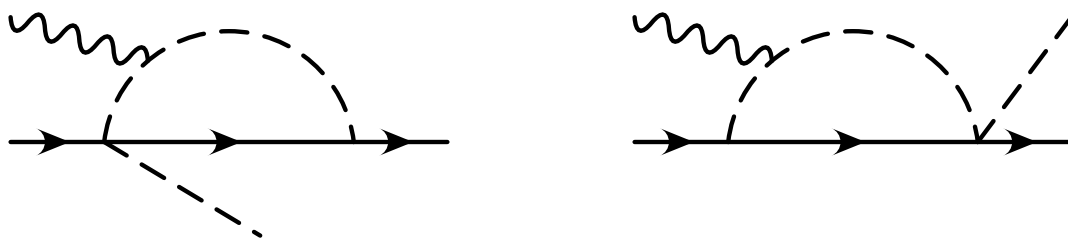


Fig. 2